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On Monoids, ... *John N. Van der Vries.*

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ON MONOIDS.

BY JOHN N. VAN DER VRIES.

With Plate XV.

The object of this paper is, first, to consider the general monoid, and, second, to classify the quartic monoids in detail.

A monoid M_a of order a is a surface of order a that has an $(a-1)$-tuple point. If the point (0, 0, 0, 1) is taken as this $(a-1)$-tuple point, the equation of the monoid can be written

$$u_a + u_{a-1} s = 0, \qquad (1)$$

where u_a and u_{a-1} are homogeneous functions of x, y and z of degrees a and $a-1$, respectively. The point (0, 0, 0, 1) is called the vertex, and the cones u_a and u_{a-1} the superior and the inferior cones, respectively, of the monoid. There are $a(a-1)$ lines on M_a passing through its vertex, viz., the lines common to the superior and inferior cones. These lines are commonly called "the lines of the monoid."* It is evident that if a line is a k-tuple line on M_a, it is a line of multiplicity k on one cone and of multiplicity not less than k on the other cone. Four kinds of lines are thus seen to be possible on the monoid, according to the relative multiplicities of the lines on the two cones of M_a. We can represent them by the following equations of monoids containing them:

I. $$(w_k z^{p+1} + w_{k+1} z^p + \ldots\ldots) + s(v_{k+1} z^{p-1} + v_{k+2} z^{p-2} + \ldots\ldots) = 0,$$

where w_k is not identically equal to 0; $v_{k+1}, \ldots\ldots v_{k+g-1}$ may be zero;

II. $$(w_k z^{p+1} + w_{k+1} z^p + \ldots\ldots) + s(v_k z^p + v_{k+1} z^{p-1} + \ldots\ldots) = 0,$$

where w_k is not identically equal to 0, and v_k does not contain w_k as a factor;

*This nomenclature is due to Cayley, who uses this surface in his consideration of twisted curves.
Comptes Rendus, t. LIV (1862), pp. 55, 396, 672.

III. $(w_{k+1}z^p + w_{k+2}z^{p-1} + \ldots\ldots) + s(v_k z^p + v_{k+1}z^{p-1} + \ldots\ldots) = 0,$

where v_k is not identically equal to 0, and w_{k-1} does not contain v_k as a factor;

IV. $(w_{k+2}z^{p-1} + w_{k+3}z^{p-2} + \ldots\ldots) + s(v_k z^p + v_{k+1}z^{p-1} + \ldots\ldots) = 0,$

where v_k is not identically equal to 0; $w_{k+2}, w_{k+3}, \ldots\ldots w_{k+g-1}$ may be zero.

In these equations, v_k, v_{k-1}, w_k, etc., represent functions of x and y of degrees k, $k+1$, k, etc., respectively; also, $k+p+1=a$. The line xy is a k-tuple line on each of these monoids, and differs in the four cases only in its relative multiplicities on the two cones. We shall hereafter distinguish between the different k-tuple lines and designate them as lines of kinds I, II, III, and IV, respectively.* No k-tuple line of kind I or II has a point on it of higher multiplicity, whereas every line of kind III or IV has on it a point of multiplicity $k+1$ (*e. g.*, the point (0, 0, 1, 0) on the monoids III and IV above). Every line of kind I, III or IV is torsal on every sheet of M_a that contains it, whereas a line of kind II is scrolar on at least one sheet of the monoid that contains it.† In the case of a line of kind II the scrolar sheets are inseparably connected and the tangent plane to them revolves through 180° as we pass along the line. The lines of kinds I, II, III and IV count as $k(k+g)$, k^2, $k(k+1)$, and $k(k+g)$ lines, respectively, of the monoid. Lines of kind III differ from lines of kind IV only in the fact that the tangent cone at a $(k+1)$-tuple point on a line of kind IV breaks up entirely into planes.

It is evident that all multiple lines on M_a pass through the vertex. The only lines that M_a can have not passing through the vertex are simple lines. For, a plane passing through the vertex and such a line of multiplicity greater than one would meet the monoid, in addition to this line, in a curve of order less than $a-1$ having an $(a-1)$-tuple point at the vertex; this is impossible. A plane through the vertex and a simple line not passing through the vertex meets the monoid, in addition to this line, in a curve of order $a-1$ having an $(a-1)$-tuple point at the vertex; that is, in $a-1$ lines passing through the vertex. Such simple lines on the monoid that do not pass through the vertex are called transversals by Cayley.‡ Thus, for every transversal on M_a it is necessary to have $a-1$ of the $a(a-1)$ lines of M_a in one plane.

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*For a fuller explanation of these lines, see a paper published in the proceedings of the American Academy of Arts and Sciences, 1902 and 1903.

†A line is torsal or scrolar on a sheet of the surface containing it according as the tangent plane to that sheet of the surface is or is not the same at every point of the line. See Cayley, Collected Papers, vol. VI, p. 334.

‡Comptes Rendus, t. LVIII (1864), pp. 994-1000.

For the same reason, it is possible to place some limits on the multiplicities of points on the monoid. We have seen that the line joining a $(k+1)$-tuple point to the vertex is at least a k-tuple line on the monoid. Suppose the monoid whose equation is (1), with vertex A (0, 0, 0, 1), has a k-tuple point B (0, 0, 1, 0) and a k'-tuple point C (0, 1, 0, 0), and the lines AB and AC as lines of kind III. Assume $2 \overline{\lessgtr} k' \overline{\lessgtr} k$. The terms of lowest degree in x, y and s will be of the form $v_{k-1}\,s + v_k$, multipled by z^{a-k}, where v_{k-1} and v_k are homogeneous functions of x and y of degrees $k-1$ and k, respectively. Similarly, the terms of lowest degree in x, z and s will be of the form $v_{k'-1}\,s + v_{k'}$, multiplied by $y^{a-k'}$, where $v_{k'-1}$ and $v_{k'}$ are homogeneous functions of x and z of degrees $k'-1$ and k', respectively. The terms of lowest degree in x and s will be of degree $k-k'$ more in y than in z, and will be of the form

$$ax^{k+k'-a}\,y^{a-k'}\,z^{a-k} + bx^{k+k'-a-1}\,y^{a-k'}\,z^{a-k}s\,;$$

the line $xs \equiv BC$ thus being a line of multiplicity $k+k'-a$ on M_a. As this cannot be a line of multiplicity greater than one, we must have

$$k+k'-a \overline{\lessgtr} 1, \text{ i. e., } k+k' \overline{\lessgtr} a+1\,;$$

that is, the monoid cannot have two points on it, excluding the vertex, the sum of whose multiplicities exceeds $a+1$. Thus, if $4 \overline{\lessgtr} a$, M_a can have at most two vertices, excluding the case where it has an infinite number and is a ruled surface. As we know, the cubic may have four, a quadric has a double infinity, and the plane a triple infinity of vertices.

If M_a has an $(a-1)$-tuple point B in addition to its vertex A, it can have in addition no point of multiplicity greater than two. The line AB is a line of multiplicity $a-2$, at least, on the monoid. (If the line is an $(a-1)$-tuple line, any point of it may be considered as the vertex of M_a. It must then be a line of kind II, and counts as $(a-1)^2$ lines of M_a. Therefore, through any point of the line taken as vertex will there pass $a(a-1)-(a-1)^2 \equiv a-1$ other lines of M_a. These are the additional lines in which the tangent planes to the $a-1$ sheets of M_a at any point of the line meet M_a.) If the line AB is an $(a-2)$-tuple line on M_a, the monoid can have no other multiple line and, therefore, no point of multiplicity greater than two. It can have a point of multiplicity two, the lines joining this point to A and B being simple lines on the surface. A plane through the $(a-1)$-tuple points A and B and a double point C intersects the surface in a curve having $(a-1)$-tuple points at A and B and a double point at C, that is, in the $(a-2)$-tuple line AB and the two simple lines AC and BC. If the $(a-2)$-tuple line AB is of kind III, it counts as $(a-2)(a-1)$ lines of M_a. The monoid can then have only $a(a-1)-(a-1)(a-2)$ or $2(a-1)$ other lines. If it has

double points, the lines joining these to the vertex must be simple lines of kind III or IV. If they are of kind III, each counts as 2 lines of M_a. The monoid can thus have at most $a-1$ double points. (If the lines are all of kind IV, each counts as β lines of the monoid; where $3 \gtreqless \beta$. M_a can have at most $\frac{2(a-1)}{\beta}$ double points.) This is also evident from the fact that a cone of order a can have at most $\frac{a(a-1)}{2}$ double edges. The $(a-2)$-tuple line on the monoid, being an $(a-1)$-tuple line on the superior cone, counts for $\frac{(a-1)(a-2)}{2}$ double edges of this cone. This cone can thus have at most $a-1$ double edges in addition to the $(a-1)$-tuple edge AB; that is, M_a can have at most $a-1$ double points in addition to the $(a-1)$-tuple point B.*

If a monoid having such an $(a-1)$-tuple point B has its maximum number of double points, these $a-1$ points all lie on one line. For the superior cone breaks up into a planes, of which $a-1$ pass through one line, the $a-1$ double edges being the intersections of these $a-1$ planes with the a-th plane of the cone. The $a-1$ double points of M_a thus lie in one plane passing through A. Similarly, considering B as the vertex, it is evident that they all lie in one plane passing through B. They must, therefore, lie on the line CD, which is the intersection of the two planes. This line CD is a simple line on M_a. A monoid, therefore, with a second $(a-1)$-tuple point and $a-1$ double points has just a transversals, $a-1$ of them being the lines from the double points to the $(a-1)$-tuple point, and the remaining one the line joining all the double points.†

If a monoid having a second $(a-1)$-tuple point, and the line from this point to the vertex as a line of kind III has $a-2$ double points in addition (that is, all but one of the maximum number of additional double points), these will in general lie on a twisted cubic; for, the superior cone, in order to have an $(a-1)$-tuple edge and $a-2$ double edges, must break up into at least $a-1$ components. If it breaks up into this number, $a-2$ of them will be planes and the remaining one a quadric cone. The $a-2$ planes will all pass through one edge of this quadric cone, the remaining lines of intersection being the $a-2$ double edges of the superior cone; *i. e.*, the lines to the $a-2$ double points of M_a. The double points will thus in general lie on two quadric cones that have the two $(a-1)$-tuple points of M_a as their vertices and the line AB as a common edge; that is, they will lie on the cubic curve that is the residual intersection of these quadric

*If the $(a-2)$-tuple line AB is of kind IV, it counts as $a(a-2)$ lines of M_a; the monoid can, therefore, have at most a other lines. It can have, however, no double points, for the superior cone contains the line AB as an a-tuple edge and can, therefore, have no double edge.

†The points will then lie as in figure (1) in the accompanying plate.

cones. In special cases, however, the superior cone from one or both vertices may break up entirely into planes. In these cases the $a-2$ double points will lie either on a conic or on a straight line.

It is thus evident that, if a monoid M_a having a second $(a-1)$-tuple point and an $(a-2)$-tuple point of kind III joining this point to the vertex has $a-m$ double points in addition, these points will in general lie on a curve of order $2m-1$ $\{$viz., $m^2-(m-1)^2\}$, which is the residual intersection of two cones of order m having different vertices but having an $(m-1)$-tuple line in common.

A plane through the $(a-2)$-tuple line and a simple line (of which there can be at most $2(a-1)$) cuts M_a in a curve of order a having $(a-1)$-tuple points at A and B. This curve consists of the $(a-2)$-tuple line AB, an ordinary line through A, and a transversal through B. This transversal intersects the simple line, if of kind II, at the point where the cutting plane coincides with the scrolar tangent plane to M_a along this line, or at the double point, if the simple line has a double point on it. A monoid with an $(a-1)$-tuple point and no double points has at least $2(a-1)$ transversals, whereas every double point reduces this minimum number by one. As we have seen, for every additional transversal (*i.e.*, additional to the $2(a-1)$ if M_a has no double points, to the $2(a-1)-\beta$ if it has β double points) $a-1$ of the lines of the monoid, excluding the $(a-2)$-tuple line, must lie in one plane. If the cones of the monoid are general cones, this does not in general occur. There are, however, special cases in which the cones break up. As both cones have a line on them of multiplicity one less than the order of the cone, at least all but one of the components of each cone are planes. If the superior cone breaks up into a planes, of which $a-1$ pass through one line, the a-th plane meets the inferior cone in $a-1$ lines, which are thus $a-1$ lines of M_a in one plane. The inferior cone can never break up entirely into planes, for the plane not passing through the multiple line would meet the superior cone in a lines; *i. e.*, we would have a lines of the monoid lying in one plane. This is impossible. A monoid thus with an $(a-1)$-tuple point B and no double points has, in general, $2(a-1)$ transversals; it may have one more.

A plane through the $(a-2)$-tuple line and not through any other line of M_a meets the monoid in the $(a-2)$-tuple line and a conic passing through both $(a-1)$-tuple points.

If M_a has an $(a-2)$-tuple point B, and if the line joining this point to the vertex is an $(a-3)$-tuple line of kind III, the monoid can have at most $2a-3$ double points in addition. The line from each of these points to B is then a simple line of kind III. If M_a has this maximum number of double points, the superior cone breaks up

entirely in planes, of which $a-2$ pass through the line AB. The double points will thus lie in two planes through the vertex, $a-1$ lying in each plane. As a plane through $a-1$ lines to $a-1$ double points intersects the monoid in addition to these lines in one other line, viz., a transversal, and as the curve of intersection of the monoid by this plane has a double point wherever the plane meets the monoid in a double point, it is evident that this transversal passes through all the double points in that plane. The $2a-3$ double points thus lie on two intersecting lines, the point of intersection being one of these $2a-3$ double points. It can be shown by similar reasoning that the $(a-2)$-tuple point lies in the same plane as these $2a-3$ double points. Thus, if $a=7$, $2a-3=11$, the points lie as in figure (2).

A monoid having an $(a-2)$-tuple point B and $2a-4$ double points will, in general, be one of two kinds. Thus, if $a=7$, $2a-4=10$, the points will lie either as in figure (3) or as in figure (4). If, as in figure (3), where the superior cone breaks up into a quadric cone, $a-3$ planes that pass through an ordinary edge of the quadric cone and a plane that does not pass through this edge, the double points all lie in one plane. If, as in figure (4), where the superior cone breaks up into a quadric cone and $a-2$ planes that have a line in common, the double points lie by twos on lines that pass through the point B.

By this method of considering the possible ways in which the cones of the monoid may break up, we can determine the possible positions of the double points on the monoid. We shall not consider any more general cases, but shall make use of this method in the consideration of the quartic monoids.

A monoid with an $(a-2)$-tuple point may have in addition a certain number of triple points. It can clearly have no points of multiplicity greater than three. A triple line on the superior cone counts for three double edges of this cone; a line to a triple point on the monoid, if it is of kind III, counts for six lines of the monoid, whereas a line to a double point counts only for two of these lines; nevertheless, we cannot always substitute a triple point for three double points. We can say that a monoid with an $(a-2)$-tuple point can never have more than $\frac{2a-3}{3}$ triple points, but the exact number must be investigated in each case.

A monoid having an $(a-k)$-tuple point on a line of kind III can have in addition points of multiplicity $k+1$ at most, where $2 \overline{\underline{<}} a-k$. If it has a $(k+1)$-tuple point C in addition to an $(a-k)$-tuple point B, the line BC is a transversal on the monoid; but if the point C is a point of any multiplicity less than $k+1$, the line BC will not

in general be a transversal. If the line AB is a line of kind III, it counts for $(a-k)(a-k-1)$ of the $a(a-1)$ lines of the monoid. The monoid with such an $(a-k)$-tuple point may thus have in addition $k(2a-k-1)$ simple lines passing through the vertex. If the line to a k-tuple point is a line of kind III, it counts as $k(k+1)$ lines of the monoid. The monoid with an $(a-k)$-tuple point can thus never have more than $\frac{2a-k-1}{k+1}$ $(k+1)$-tuple points.

A monoid can have at most $\frac{a(a-1)}{2}$ double points, for every line of kind III joining a double point of a monoid to its vertex counts for two lines of the monoid. This is also evident from the fact that this is the maximum number of double lines possible on the superior cone; for we can always pass a cone of order $a-1$ through this number of lines, and can therefore always have these double lines of the superior cone as simple lines on the inferior cone, and, therefore, as simple lines of kind III on the monoid. If the monoid has this maximum number of double points, the superior cone breaks up entirely into planes through the vertex. Each of these a planes meets each of the remaining $a-1$ planes in a line that is a line of the monoid to a double point. Thus $a-1$ double points lie in each of these a planes. Each plane thus intersects the monoid in $a-1$ lines through the vertex and in a transversal which passes through all the double points in the cutting plane. The $\frac{a(a-1)}{2}$ double points thus lie on a lines, $a-1$ of them on each line, *and therefore lie in one plane.* If $a=7$, the points will lie as in figure (5).

No monoid can have on it a multiple curve of an order higher than the first, and then, as we have seen, it passes through the vertex; for, any line drawn from the vertex to a k-tuple curve, say of order β, where $2 \overline{\gtrless} k$, would meet the monoid in at least $a+1$ points, and would therefore lie on the monoid. The monoid would then break up into a cone of order β, having the vertex of the monoid as vertex and the curve as base, and a monoid of order $a-\beta$.

Classification of Quartic Monoids.

1. The equation of the general quartic monoid with vertex at the point A(0, 0, 0, 1) may be written

$$u_4 + u_3 s = 0,$$

where u_4 and u_3 are homogeneous functions of x, y and z of degrees 4 and 3, respectively. This monoid has twelve lines on it passing through the vertex; it will, in addition, have a transversal whenever three of these twelve lines lie in one plane.

2. If M_4 has another triple point (say the point B(0, 0, 1, 0), and if

the line AB is a line of kind III, the equation of the monoid may be written

$$u_4 + (az + bs)u_3 + u_2zs = 0, \qquad (2)$$

where u_4, u_3 and u_2 are homogeneous functions of x and y of degrees 4, 3, and 2, respectively. This line $AB \equiv xy$ is a double line on M_4, but counts as six lines of the monoid. This monoid thus has in general six other lines. A plane through the double line AB and one of these six lines meets the monoid in addition in a transversal that intersects AB at B and the single line at the point where the tangent plane to the surface along this line coincides with the cutting plane. There will thus be six transversals on the monoid, all passing through the point B. For every additional transversal it is necessary for the two cones of the monoid to have three of their six simple intersections in one plane. If the cones are proper cones—that is, in the general case of the monoid whose equation is (2)—this does not occur. The monoid then has six lines in addition to AB passing through each vertex, each line through one vertex meeting one line through the other vertex. There will be no other lines on this monoid.

If, however, the superior cone breaks up into four planes, of which three pass through the line xy, while the inferior cone does not break up, there will be three lines of the monoid in one plane, viz., the intersections of the inferior cone by the fourth plane of the superior cone. A cross-section of the two cones by a plane not passing through their common vertex will show this. See figure (6).* Taking x, y, $ax + by$ and z as the four planes of the superior cone, the equation of the monoid will be of the form

$$(ax + by)xyz + s(u_3 + u_2z) = 0,$$

where u_3 and u_2 are homogeneous functions of x and y of degrees 3 and 2, respectively. This monoid will thus have seven transversals, or thirteen lines in all, in addition to the double line AB. In figure (6), 1, 2 and 3 are the traces of lines that lie in one plane. It is, however, possible that the lines whose traces are 4, 5 and 6 also lie in one plane; the monoid then has eight transversals. Similarly, when the superior cone breaks up into planes and the inferior cone breaks up into a plane and a quadric cone, the monoid has either seven or eight transversals. The equation may then be written

$$(ax + by)xyz + su_1u_2 = 0,$$

where u_1 and u_2 are homogeneous functions of x, y and z of degrees 1 and 2, respectively. The traces of the lines will then be as in figure (7). The lines whose traces are 1, 2 and 3 lie in one plane; the line

*In the figures representing the traces of the cones of the monoid on a plane, we use broken lines to distinguish the inferior from the superior cone.

whose trace is 2 may or may not lie in the same plane with the lines whose traces are 4 and 5.

A plane passing through the line AB but not through one of these six ordinary* lines of the monoid cuts the surface, in addition to the double line AB, in a conic that passes through both points A and B.†

3. A quartic monoid may have a double point, say the point $C(0, 1, 0, 0)$, in addition to the two triple points A and B. If the line AC is a simple line of kind III, it will count for two lines of the monoid. The equation may then be put into the form

$$x^2u_2 + xzv_2 + xsw_2 + zst_2 = 0,$$

where u_2, v_2, w_2 and t_2 are homogeneous functions of x and y of degree 2. The superior cone breaks up into the plane x and a cubic cone that contains the line xy as a double line. In addition to the line xy and the simple line xz, there are four ordinary lines on the monoid. A plane through the double line and one of the five simple lines cuts the monoid in addition in a transversal that intersects the double line in the triple point and the simple line either at the double point or at the point where the tangent plane coincides with this cutting plane. This monoid thus has in general five transversals. There may be cases, as in the previous monoid, where three of the five simple lines lie in one plane, thus causing an additional transversal.

4. If M_4 has a second double point, say the point $D(1, 0, 0, 0)$, and if the line from this point to the vertex is a line of kind III, the equation may be written

$$x^2y^2 + xyzu_1 + xysv_1 + u_2zs = 0,$$

where u_1, v_1 and v_2 are homogeneous functions of x and y of degrees 1, 1, and 2, respectively. The superior cone breaks up into two planes x and y and a quadric cone that contains the line xy as a simple edge. There are through each triple point two ordinary lines in addition to the two lines to the double points. This monoid thus has in general four transversals passing through the second triple point. As in the previous cases, three of the four simple lines may lie in one plane, thus causing an additional transversal. It is evident that such a line may be considered as a transversal with respect to both vertices.

5. A quartic monoid can have at most three double points in addition to a second triple point. The lines from the three double points to either triple point are simple lines of the monoid, and are lines of kind III; each one, therefore, counts as two lines of the monoid, considering the triple point through which it is drawn as vertex. Thus,

* Hereafter, when we speak of ordinary lines, we mean simple lines of kind II.

† In every case where the monoid has two triple points, either point may be taken as the vertex; whatever is true for the lines passing through one triple point is likewise true for the lines passing through the other triple point.

through either vertex there is a double line to the other vertex and three simple lines to the three double points. The superior cone, in order to have a triple edge and three double edges, must break up into four planes, three of which pass through the line xy. It is thus evident that the three double points must lie on one straight line. The monoid thus has four transversals; it can have no more. If the four planes of the superior cone are the planes x, y, z, and $ax + by$, the equation of the monoid may be written

$$(ax + by)(cz + ds)xy + zsu_2 = 0,$$

where $x = z = s = 0$, $y = z = s = 0$ and $ax + by = z = s = 0$ are the three double points.

6. If, in addition to the triple point B on the line AB of kind III, the monoid has a double point C(0, 1, 0, 0) lying on a line AC of kind IV, where the line AC is a simple line on the inferior cone and a triple line on the superior cone, the equation of the monoid may be written

$$x^3u_1 + x^2zv_1 + xsw_2 + zst_2 = 0,$$

where u_1, v_1, w_2 and t_2 are homogeneous functions of x and y of degrees 1, 1, 2, and 2, respectively. The superior cone breaks up into the plane x doubled and a quadric cone that contains the lines xy and xz as simple lines. The line xz thus counts for three of the six lines of the monoid. The line BC also counts for three lines of the monoid when B is considered as the vertex. The monoid has in general three transversals, the transversals being the additional lines in which the planes through the line AB and the three lines through the vertex cut the monoid. It has an additional transversal only if the three lines lie in one plane, or if two of them lie in one plane with the line AC; in the latter case the transversal passes through the point C.

7. If the monoid has in addition a second double point, say the point D(1, 0, 0, 0), the line AD will be a simple line of kind III on the monoid. The equation of the monoid may then be written

$$x^2yz + xysu_1 + zsu_2 = 0,$$

where u_1 and u_2 are homogeneous functions of x and y of degrees 1 and 2, respectively. There is thus one ordinary line passing through A in addition to the lines AB, AC, and AD; it lies in the same plane with AC and AD. For the plane $z = 0$ meets the monoid in the lines AC, AD, CD, and $z \quad u_1 = 0$. Call this last line AE, where E is the point in which it meets CD. This quartic monoid therefore has on it four lines, viz., AB, AC, AD, and AE, and four transversals, viz., BC, BD, BE, and CD. It can have no other line.

8. If the triple point B is connected with the vertex A by a line of kind IV, the equation of the monoid must be of the form

$$u_4 + u_3s + u_2zs = 0,$$

where u_4, u_3 and u_2 are homogeneous functions of x and y of degrees 4, 3, and 2, respectively. The line xy is a quadruple line on the superior and a double line on the inferior cone; it therefore counts as eight of the twelve lines of the monoid. The monoid, therefore, always has four lines on it passing through each triple point, every line passing through one of the two triple points meeting one line through the other triple point. If three of the lines through one triple point lie in one plane, the three corresponding lines through the other triple point lie in one plane; the monoid then has an additional transversal. A plane through the double line AB, and not through an ordinary line of this M_4, meets the monoid in addition in a conic that passes through the two triple points.

9. If the monoid has in addition a double point C(0, 1, 0, 0) on a line AC of kind III, the equation may be written

$$x^2u_2 + xsv_2 + zsw_2 = 0,$$

where u_2, v_2 and w_2 are homogeneous functions of x and y of degree 2. Since the line AC counts for two lines of M_4, there are only two other lines on M_4 through A. The monoid has in general three transversals —one the line BC, and the other two lines that pass through the triple point and intersect the two ordinary lines through A at the points where the tangent planes to the monoid along these lines coincide with the cutting planes. If we take these two ordinary lines to be the lines yz and (ax + by)z, the equation of the monoid may be written

$$xy(ex + fs)(ax + by) + xzs(cx + dy) + y^2zs = 0.$$

The line $z = ex + fs = 0$ lies on this monoid; M_4 thus has four transversals in this case.

10. If the monoid has in addition a second double point, say the point D(1, 0, 0, 0), that lies on a line AD of kind III, the equation may be written

$$x^2y^2 + xysu_1 + zsu_2 = 0,$$

where u_1 and u_2 are homogeneous functions of degrees 1 and 2, respectively. The monoid has no lines in addition to the lines AB, AC, and AD. It has two transversals, viz., the lines BC and BD; it can have no other. A plane through the two lines AC and AD intersects the monoid in addition in a conic that passes through the vertex A and the two double points C and D.

11. If the monoid that has a triple point B lying on a line AB of kind IV has, in addition, a double point C on a line AC of kind IV, it may be of two kinds. The line AC may be a triple line on the superior and a simple line on the inferior cone. The equation may then be written

$$x^3u_1 + xsu_2 + zsv_2 = 0,$$

where u_1, u_2 and v_2 are homogeneous functions of x and y of degrees 1, 2, and 2, respectively. The line AC counts for three lines of M_4; the monoid can therefore have but one additional line through A. There are, therefore, only two transversals, one being the line BC and the other the additional line in which a plane through AC and the ordinary line through A intersects the monoid. If, however, the line AC is a quadruple line on the superior cone and a simple line on the inferior cone, the equation may be written

$$x^4 + xsu_2 + zsv_2 = 0,$$

where u_2 and v_2 are homogeneous functions of x and y of degree 2. The line AC then counts as four lines of the monoid; the monoid has, therefore, no more lines through A. This monoid has thus but one transversal, viz., the line BC.

12. A monoid M_4 that has no second triple point can have at most six double points; for every double point causes a line on the monoid that counts for at least two lines of the monoid. It is thus evident that, if a quartic monoid has six double points, the lines joining these points to the vertex are all lines of kind III on the monoid. If the monoid has β double points (where $\beta \gtreqless 6$), and if the lines from these to the vertex are all lines of kind III, the monoid may have β lines from the vertex to the β double points and $12 - 2\beta$ ordinary lines through the vertex, or $12 - \beta$ lines in all. There are thus six different quartic monoids that have only double points and that have these joined to the vertex by lines of kind III.

In general, a plane passing through two lines on a monoid intersects the monoid, in addition, in a conic that passes through the vertex and that crosses the lines at the double points on them, if there are any, otherwise at the points at which the tangent plane coincides with the cutting plane. In special cases, however, this conic may break up into a line through the vertex and a transversal.

If the quartic monoid has a double point, say the point B(0, 0, 1, 0), the equation of the monoid may be written

$$u_4 + u_3z + u_2z^2 + v_3s + v_2zs + v_1z^2s = 0,$$

where u_4, u_3, u_2, v_3, v_2 and v_1 are homogeneous functions of x and y of degrees 4, 3, 2, 3, 2, and 1, respectively. The line AB then counts for two of the twelve lines of the monoid; the monoid thus has, in addition, ten ordinary lines passing through the vertex. It will not in general have any transversals.

If the quartic monoid has a second double point, say the point C(0, 1, 0, 0), the equation may be written

$$x^2u_2 + xzv_2 + z^2w_2 + xst_2 + zsr_2 + z^2su_1 = 0,$$

where u_2, v_2, w_2, t_2, r_2 and u_1 are homogeneous functions of x and y of

degrees 2, 2, 2, 2, 2, and 1, respectively. The line AC also counts for two lines of the monoid; the monoid, therefore, has eight ordinary lines through the vertex in addition to the lines AB and AC; that is, ten lines in all. It will not in general have any transversals.

If the quartic monoid has a third double point, say the point D(1, 0, 0, 0), the equation of the monoid may be written

$$x^2y^2 + xyzu_1 + z^2u_2 + xysv_1 + zsv_2 + z^2sw_1 = 0,$$

where u_1, u_2, v_1, v_2 and w_1 are homogeneous functions of x and y of degrees 1, 2, 1, 2, and 1, respectively. This monoid has six ordinary lines through the vertex in addition to the lines AB, AC, and AD; that is, nine lines in all. This monoid does not in general have the three double points lying in one plane. If, however, we take these three points to be the points $x = y = s = 0$, $x = z = s = 0$, and $x = y + z = s = 0$, the superior cone breaks up into the plane x and a cubic cone. The plane x meets the inferior cone in three lines, which are the lines of kind III, to the three double points. The equation of the monoid may then be written

$$x^2u_2 + x^2zu_1 + xsv_2 + xzsv_1 + xz^2t_1 + yz(y + z)w_1 = 0,$$

where u_2, u_1, v_2 and v_1 are homogeneous functions of x and y of degrees 2, 1, 2, and 1, respectively, and where t_1 and w_1 are homogeneous functions of x and s of degree 1. The plane x intersects the monoid in the four lines $x = y = 0$, $x = z = 0$, $x = y + z = 0$, $x = s = 0$; the monoid thus has a transversal passing through the three double points.

The equation of the monoid that has four double points, and that has these points joined to the vertex by lines of kind III, is of the form

$$u_2v_2 + u_3s = 0,$$

where u_2, v_2 and u_3 are homogeneous functions of x, y and z of degrees 2, 2, and 3, respectively. The superior cone breaks up into two quadric cones, the lines of intersection of these cones being simple lines on the inferior cone. If these cones are general quadric cones, no three of the four lines to double points will lie in one plane. Two of them may, however, lie in the same plane as one of the four additional lines through the vertex; the monoid will then have an additional transversal through the two double points in that plane. It will also have an additional transversal if three of these four lines lie in one plane.

If a quartic monoid has five double points, and has the lines from these points to the vertex as lines of kind III, its equation may be written

$$u_1v_1w_2 + u_3s = 0,$$

where u_1, v_1, w_2 and u_3 are homogeneous functions of x, y and z of degrees 1, 1, 2, and 3, respectively. The superior cone thus breaks up

into a quadric cone and two planes; it therefore has five double edges which are simple lines on the inferior cone. The monoid contains only two ordinary lines in addition to the lines to these five double points. It is evident, from the way in which the superior cone breaks up, that there are always two planes that contain three lines to double points. There are, therefore, always two transversals on a quartic monoid of this kind, and each of these two lines passes through three double points. The five double points therefore lie in one plane, as in figure (8). There may be additional cases in which one of the two ordinary lines of kind II lies in a plane with two of the lines to double points, or in which the two lines lie in a plane with one of the lines to a double point, an additional transversal being the result in each case.

The equation of the quartic monoid that has six double points, and has these points joined to the vertex by lines of kind III, may be written in the form

$$u_1v_1w_1t_1 + u_3s = 0,$$

where u_1, v_1, w_1, t_1, and u_3 are homogeneous functions of x, y and z of grees 1, 1, 1, 1, and 3, respectively. The superior cone breaks up into four planes, for in no other way can it have six double edges; the inferior cone contains these lines as simple edges. These six lines lie by threes in four planes. There are, therefore, four transversals, each one passing through three double points. The six double points therefore lie in one plane. The monoid can have no line on it in addition to the six to the double points and no transversal in addition to the four joining the six double points.

13. If the quartic monoid has a double point, say the point B(0, 0, 1, 0), and has the line AB as a particular line of kind IV (viz., as a line of multiplicity three on the superior cone and of multiplicity one on the inferior cone), its equation may be written

$$u_4 + u_3z + v_3s + u_2zs + u_1z^2s = 0, \qquad (3)$$

where u_4, u_3, v_3, u_2 and u_1 are homogeneous functions of x and y of degrees 4, 3, 3, 2, and 1, respectively. The monoid may have nine ordinary lines in addition to the line AB. It will not in general have a transversal.

If the monoid whose equation is (3) has a second double point, say the point C(0, 1, 0, 0), and has the line AC as a simple line of kind III, its equation may be written in the form

$$x^2u_2 + xzv_2 + xsw_2 + zst_2 + z^2su_1 = 0,$$

where u_2, v_2, w_2, t_2 and u_1 are homogeneous functions of x and y of degrees 2, 2, 2, 2, and 1, respectively. The monoid has seven lines passing through the vertex in addition to the lines AB and AC. The superior cone breaks up into the plane x and a cubic cone. The

plane x intersects the monoid in the lines AB, AC, BC, and $(x = ay + bz = 0)$. The monoid thus has in general one transversal; it may in special cases have more, as in some of the monoids already considered.

If this quartic monoid has a third double point, say the point $D(1, 0, 0, 0)$, and if the line AD is a line of kind III on the monoid, the equation of the monoid may be written

$$x^2y^2 + xyzu_1 + xysv_1 + zsu_2 + z^2sw_1 = 0,$$

where u_1, v_1, u_2 and w_1 are homogeneous functions of x and y of degrees 1, 1, 2, and 1, respectively. The superior cone breaks up into two planes and a quadric cone. Each of these two planes intersects the inferior cone of the monoid in three lines that are therefore three lines of the monoid lying in one plane. The monoid therefore has, in general, two transversals, each one passing through two double points. The monoid has on it five ordinary lines in addition to the lines AB, AC, and AD.

If this monoid has a fourth double point, say the point E, and if the line AE is a line of kind III on the monoid, the superior cone must break up entirely into planes, of which three pass through the line $AB \equiv xy$. The three lines AC, AD and AE are therefore in one plane. We can therefore take the line AE to be the line $ax + by = z = 0$. The plane z intersects the monoid in the three lines AC, AD, and AE, and in a transversal that passes through the three double points C, D, and E. The equation of the monoid may then be written

$$xyz(ax + by) + xys(ax + by) + zsu_2 + z^2su_1 = 0,$$

where u_2 and u_1 are homogeneous functions of x and y of degrees 2 and 1, respectively, Each of the three planes through the triple line of the superior cone intersects the inferior cone in three lines that are lines of the monoid lying in one plane. This monoid, therefore, always has four transversals, one of which passes through three double points and the other three each through two double points. This monoid has on it three ordinary lines in addition to the four lines to the double points.*

This monoid cannot have any more double points, as the superior cone cannot break up into any more components, three double edges being the maximum number of double edges that a quartic cone can have in addition to a triple edge.

14. If the quartic monoid whose equation is (3) has a second double point, say the point $C(0, 1, 0, 0)$, and has the line AC as a

*There is a special monoid with four double points that has only three transversals, viz., the monoid whose superior cone breaks up into a double plane and two single planes that do not have a line of the double plane in common, where three of the lines to double points are lines of the double plane of the superior cone, as in figure (9).

triple line on the superior cone and a simple line on the inferior cone, the equation of the monoid may be written

$$x^3u_1 + x^2zv_1 + xsu_2 + zsv_2 + z^2sw_1 = 0, \qquad (4)$$

where u_1, v_1, u_2, v_2 and w_1 are homogeneous functions of x and y of degrees 1, 1, 2, 2, and 1, respectively. The superior cone breaks up into a double plane and two single planes, the lines common to the double plane and the single planes being simple lines on the inferior cone. Each of the three planes of the superior cone meets the inferior cone in three lines, which are, therefore, lines on the monoid; this monoid therefore has, in general, three transversals. There are six ordinary lines on the monoid in addition to the lines AB and AC.

The monoid whose equation is (4) cannot have another double point on it and have the line from this point to the vertex as a triple line on the superior cone and a simple line on the inferior cone. It may, however, have a double point, say the point D(1, 0, 0, 0), and the line AD as a simple line of kind III; its equation may then be written

$$x^2yz + xysu_1 + zsu_2 + z^2sv_1 = 0, \qquad (5)$$

where u_1, u_2 and v_1 are homogeneous functions of x and y of degrees 1, 2, and 1, respectively. The superior cone breaks up into a double plane and two single planes, as in the previous case; the intersection of the two single planes being, moreover, an edge of the inferior cone of this monoid. The double plane of the superior cone intersects the inferior cone in three lines, of which two are the lines AB and AC, and of which the third, being a double line on the superior cone and a single line on the inferior cone, is also a line to a double point. Therefore, if the monoid whose equation is (4) has one double point in addition it must have a second double point, and each of these must be connected with the vertex by a line of kind III. This monoid whose equation is (5) has therefore, in general, three transversals, two of which pass through two and the third of which passes through three double points. This monoid has two ordinary lines in addition to the lines to the double points, and they lie in the two planes determined by the vertex and the two transversals that pass through only two double points. The double points will lie as in figure (10).

The monoid cannot have any more double points, as the superior cone cannot break up into any more components, one double edge being the maximum number of double edges that a quartic cone can have in addition to two triple edges.

15. If the quartic monoid has a double point, say the point B(0, 0, 1, 0), and has the line AB as a quadruple line on the superior cone and a simple line on the inferior cone, it is evident that the su-

perior cone must break up entirely into planes that pass through the line AB. The equation of the monoid may then be written

$$u_4 + u_3 s + u_2 z s + u_1 z^2 s + z^3 s = 0, \qquad (6)$$

where u_4, u_3, u_2 and u_1 are homogeneous functions of x and y of degrees 4, 3, 2, and 1, respectively. Each of the planes of the superior cone meets the inferior cone in two lines in addition to the line to the double point; the monoid, therefore, always has four transversals passing through the double point. There are eight ordinary lines on the monoid in addition to the line AB; they lie by twos in four planes that pass through the line AB. If three lines of which no two lie in one of these four planes lie in one plane, an additional transversal will lie on the monoid.

If this monoid has in addition a double point, say the point C(0, 1, 0, 0), and has the line AC as a simple line of kind III, the equation of the monoid may be written

$$x^2 u_2 + x s v_2 + z s w_2 + z^2 s u_1 + z^3 s = 0,$$

where u_2, v_2, w_2 and u_1 are homogeneous functions of x and y of degrees 2, 2, 2, and 1, respectively. The superior cone breaks up into a double plane and a quadric cone; the double plane intersects the inferior cone in three lines, of which two are the lines AB and AC, and of which the third, being a double line on the superior cone and a single line on the inferior cone, is also a line to a double point. Therefore, if the monoid whose equation is (6) has one double point in addition, it must have a second double point, and each of these must be connected with the vertex by a line of kind III. There are four ordinary lines on the monoid in addition to the lines to the double points, and they lie by twos in the two planes which are single components of the superior cone. This monoid always has three transversals, of which one passes through all the double points.

If the monoid whose equation is (6) has in addition a double point C(0, 1, 0, 0), and has the line AC as a line of kind IV that is a triple line on the superior cone and a simple line on the inferior cone, its equation may be written

$$x^3 u_1 + x s v_2 + z s w_2 + z^2 s v_1 + z^3 s = 0,$$

where u_1, v_2, w_2 and v_1 are homogeneous functions of x and y of degrees 1, 2, 2, and 1, respectively. The superior cone breaks up into the triple plane x and another plane through the line AB; the triple plane meets the inferior cone in three lines, of which two are the lines AB and AC, and of which the third, being a triple line on the superior cone and a simple line on the inferior cone, is a line to a double point that counts for three lines of the monoid. The monoid thus has on it two transversals, of which one passes through all three double

points and the other one through only one double point. The monoid has on it two ordinary lines in addition to the lines to the three double points, and these two lines lie in the plane that is the simple component of the superior cone.

If the monoid whose equation is (6) has in addition a double point C(0, 1, 0, 0), and has the line AC as a line of kind IV that is a quadruple line on the superior cone and a simple line on the inferior cone, its equation may be written

$$x^4 + xsu_2 + zsv_2 + z^2su_1 + z^3s = 0,$$

where u_2, v_2 and u_1 are homogeneous functions of x and y of degrees 2, 2, and 1, respectively. The superior cone breaks up into the quadruple plane x. This plane intersects the inferior cone in three lines, of which two are the lines AB and AC, and of which the third, being a quadruple line on the superior cone and an ordinary line on the inferior cone, is a line to a double point that counts for four lines of the monoid. The monoid has on it no lines passing through the vertex in addition to the lines to the three double points. It has on it only one transversal, and this passes through the three double points.

16. We have finished all possible cases of quartic monoids having on them only multiple points that lie on lines of multiplicity one less than the multiplicity of the lines; *i. e.*, all possible cases of monoids that have on them only lines of kinds III and IV in addition to simple lines of kind II. We shall now consider the possible cases of quartic monoids that have lines of kind I and multiple lines of kind II in addition to these multiple points that lie on lines of kinds III and IV. It is evident that a quartic monoid that has a second triple point can have on it in addition no multiple line of kind I or II.

(*To be continued.*)
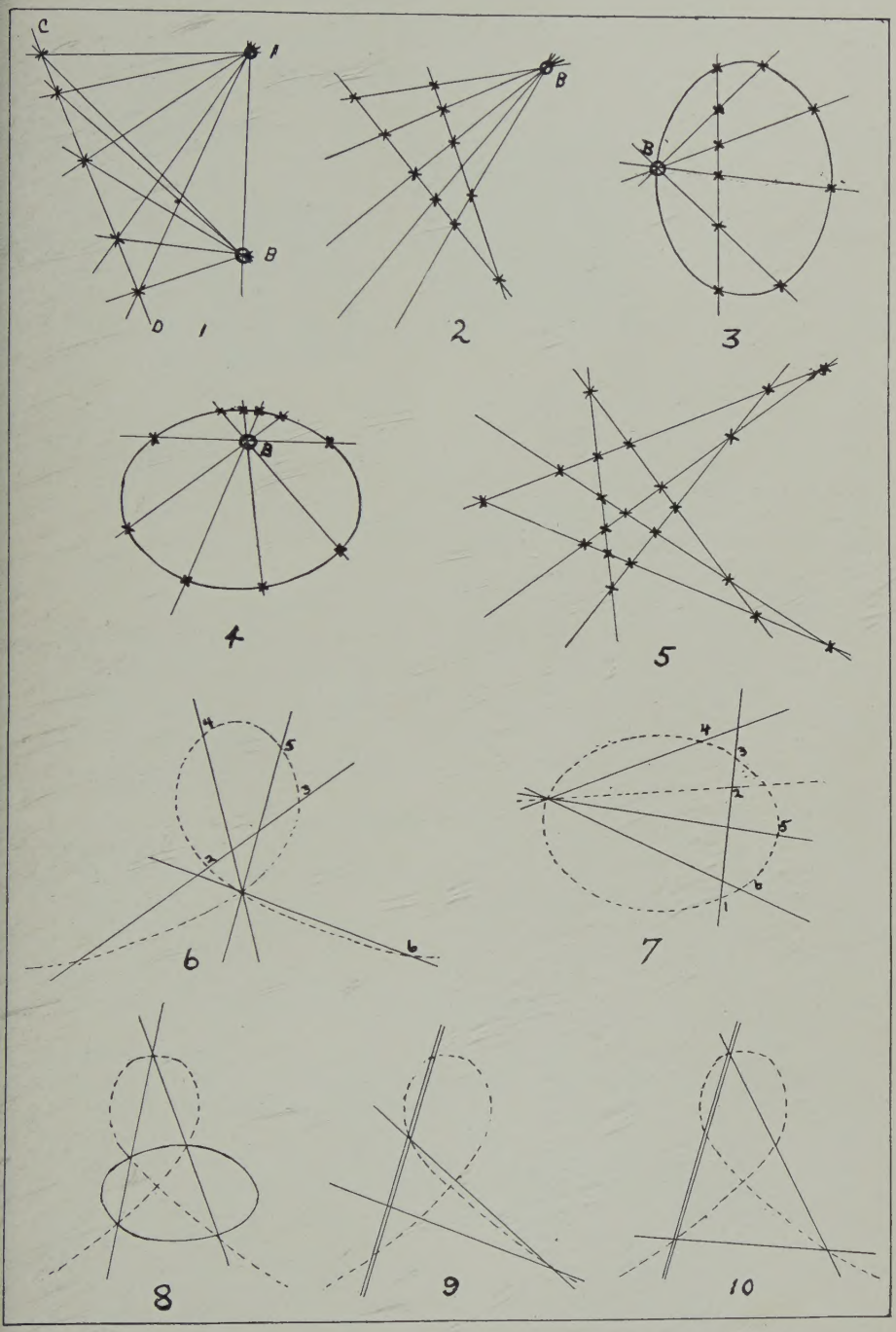
PLATE XV.